What Magnetar Seismology can Teach us about the Magnetic Fields

Rashid Shaisultanov[2], David Eichler [1]

ABSTRACT

The effect of magnetic fields on the frequencies of toroidal oscillations of neutron stars is derived to lowest order. Interpreting the fine structure in the QPO power spectrum of magnetars following giant flares reported by Strohmayer and Watts (2006) to be "Zeeman splitting" of degenerate toroidal modes, we estimate a crustal magnetic field of order 10^{15} Gauss or more. We suggest that residual m, -m symmetry following such splitting might allow beating of individual frequency components that is slow enough to be observed.

Subject headings: stars: magnetic—pulsar: neutron—stars: oscillations—X-rays: stars

The discovery of quasi-periodic oscillations (QPO) in the hyperflares of soft gamma-ray repeaters (Israel et al. 2005; Strohmayer & Watts 2005; Watts & Strohmayer 2006; Strohmayer & Watts 2006) has attracted much attention to the study of nonradial oscillations of neutron stars with solid crust. These oscillations were studied extensively in (Hansen & Cioffi 1980; McDermott, van Horn & Hansen 1988) for non-magnetic case. Unno et al. (1989) discuss in their book many aspects of the theory of nonradial oscillations of stars. The properties of nonradial modes of strongly magnetized neutron stars have been investigated by several authors (Duncan 1998; Piro 2005; Lee 2007, 2008; Sotani et al. 2006, 2007). Their main focus was on the study of axisymmetric modes. Duncan anticipated toroidal oscillations as a result of giant flares and noted that the magnetic field could affect their frequencies. The effects of a strong vertical magnetic field on the oscillation spectrum of a cylindrical slab model were studied in (Carroll et al. 1986). The nonradial modes are generally divided into two main classes: the spheroidal and toroidal modes. We will concentrate on the study of toroidal modes because of their possible connection with QPOs. They are defined by conditions

[1]Physics Department, Ben-Gurion University, Be'er-Sheva 84105, Israel; eichler@bgu.ac.il

[2]Physics Department, Ben-Gurion University, Be'er-Sheva 84105, Israel; rashids@bgu.ac.il

$$\vec{\nabla} \cdot \vec{u} = 0 \, , u_r = 0 \tag{1}$$

where $\vec{u}$ is a displacement vector.

Without magnetic field and rotation, toroidal modes (denoted ${}_lt_n$, where the index n is the number of radial nodes in the eigenfunction) have frequencies that do not depend on m. This degeneracy is lifted by magnetic field since it breaks the spherical symmetry of the problem. In this paper we will study how this happens considering the magnetic field $\vec{B}$ as a perturbation. The influence of a magnetic perturbation on spheroidal modes was considered earlier [see e.g. (Unno et al. 1989)]. We also refer to Dahlen & Tromp (1998) for a detailed discussion of perturbation theory with applications in seismology. Assuming an oscillatory time dependence $\vec{u} \propto e^{-i\omega t}$, where ω is the mode frequency, the equation for eigenfunctions without a magnetic field is

$$-\, \omega^2 \vec{u} = A(\vec{u}) \tag{2}$$

where the linear operator A describes the dynamics given specific parameters for the neutron star. We do not need its exact form here. Interested readers may consult e.g. (Hansen & Cioffi 1980; McDermott, van Horn & Hansen 1988). Let us denote toroidal eigenfunctions and frequencies obtained from this equation as $\vec{u}^{(0)}_{lm}$ and $\omega^{(0)}_l$. In spherical coordinates components of $\vec{u}^{(0)}_{lm}$ are

$$u^{(0)}_\theta = \frac{w^{(0)}_l(r)}{\sin\theta} \frac{\partial Y_{lm}}{\partial \varphi} \, , u^{(0)}_\varphi = -w^{(0)}_l(r) \frac{\partial Y_{lm}}{\partial \theta}. \tag{3}$$

Here $w^{(0)}_l(r)$ is a radial eigenfunction.

With a magnetic field we have

$$-\, \omega^2 \vec{u} = A(\vec{u}) + \frac{1}{4\pi\rho} [(\vec{\nabla} \times \vec{b}) \times \vec{B}] \tag{4}$$

where

$$\vec{b} = \vec{\nabla} \times (\vec{u} \times \vec{B}) \tag{5}$$

We now take $\vec{B}$ to be the uniform field

$$\vec{B} = B_0 \vec{e}_z. \tag{6}$$

It is helpful to use vector spherical harmonics [see e.g. (Varshalovich et al. 1988)] to express the vector operators in equations (4) and (5) in Y_{lm} representation. Definitions of vector spherical harmonics and some useful formulae are given in the appendix. Then, using the perturbation approach described in Unno et al. (1989), we obtain

$$\omega_{lm} = \omega^{(0)}_l + \omega^{(1)}_{lm} \tag{7}$$

where

$$\frac{\omega_{lm}^{(1)}}{\omega_l^{(0)}} = -\frac{B_0^2}{8\pi\omega_l^{(0)2}}\frac{1}{\int\rho w_l^{(0)2}r^2dr}\int r^2drw_l^{(0)}\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dw_l^{(0)}}{dr}\right)-\frac{l\left(l+1\right)}{r^2}w_l^{(0)}\right]F\left(l,m\right) \quad (8)$$

where

$$F\left(l,m\right)=\frac{l\left(l+2\right)\left(\left(l+1\right)^2-m^2\right)}{\left(l+1\right)^2\left(2l+1\right)\left(2l+3\right)}+\frac{m^2}{l^2\left(l+1\right)^2}+\frac{\left(l^2-1\right)\left(l^2-m^2\right)}{l^2\left(2l+1\right)\left(2l-1\right)} \quad (9)$$

Table 1 lists values of $F\left(l,m\right)$ for specific modes.

This is the main result of our paper within the framework of a perturbative calculation. Perturbation theory is applicable when

$$\frac{\omega_{lm}^{(1)}}{\omega_l^{(0)}} \ll 1 \quad (10)$$

For the case of a spherical star with uniform shear modulus μ, one can calculate the integral in (8) exactly and obtain

$$\frac{\omega_{lm}^{(1)}}{\omega_l^{(0)}} = \frac{B_0^2}{8\pi\mu}F\left(l,m\right) \quad (11)$$

By (10) perturbation theory is thus applicable when

$$\frac{B_0^2}{8\pi\mu} \ll 1 \quad (12)$$

Because of the axisymmetry of the magnetic field the (2l+1) degeneracy in m is only partially lifted : it depends only on $|m|$. The mode splits into (l+1) modes.

The "92 hertz" mode in the QPO data following the 2004 giant flare of SGR 1806-20 actually seems to be accompanied by a conglomerate of many peaks in the power spectrum, ranging from about 78 to 105 hz [Strohmayer & Watts (2006), Watts & Strohmayer (2006)], implying that at it most extreme value $\frac{\omega_{lm}^{(1)}}{\omega_l^{(0)}} \sim 0.25$. The 92 hertz QPO is attributed to the l=7 toroidal mode [Strohmayer & Watts (2006), Watts & Strohmayer (2006)]. There also seems to be a significant component at about 80 hz [panels 1-4, and 17 of figure 9 in Strohmayer & Watts (2006)], which could be the l=6 mode. [If the field is axisymmetric and has mirror symmetry around the equator, then the mechanism for luminosity variation proposed by Timokhin, Eichler & Lyubarsky (2007) works for odd l modes, where the two foot points of a given magnetic field line move in opposite directions. This introduces a twist in the magnetic field line that implies a current perturbation. In the more likely situation that the field lacks this high degree of symmetry, even l modes are also possible.] However,

there are also modes in the 80-90 hz range and also significantly above 92, up to 100 hz or more, so it is hard to interpret the data unambiguously. While there is some evidence for systematic increase of the frequency with time, the QPO signal in any given time frame appears to be constant, and in some cases several bands appear simultaneously, separated by several hz. We conclude that if the splitting is due to a magnetic field, it is at least 2 percent, and at most about 25 percent.

According to the calculations here, the variation is expected to be of order 0.4 to $0.5\frac{B_0^2}{8\pi\mu}$, as m ranges from 0 to 7. This implies that $\frac{B_0^2}{8\pi\mu} \sim 0.04$, or $B \sim 0.2\sqrt{8\pi\mu}$ if we interpret the 2 percent splitting to be due to magnetic effects.

If the entire range of from 78 to 105 hz is attributed to magnetic splitting, our first order approximation is only marginal at this strength, and a higher order calculation would give a slightly lower value for the frequency shift by a given field strength, so all we can say is that the field is of the same order as $(8\pi\mu)^{1/2}$.

The value of $(4\pi\mu)^{1/2}$ has been estimated by to be about 6×10^{15} Gauss (Thompson, C. & Duncan, R. C (1995)), so, if we assume a magnetic splitting of 2 percent, the value of B appears to be of order 1.7×10^{15} Gauss or higher, in reasonably good agreement with that estimated for the dipole field component, 1.6×10^{15} (Woods et al. (2002); Palmer et al. (2005)). Note that in the geometry used here, the field lines do not particularly lie within the crust but rather cut through it vertically at angle $\pi/2-\theta$, where θ is the latitude, so a purely toroidal field would be estimated to be somewhat weaker than the above estimate by a factor of 30 percent or so. On the other hand, a toroidal field could easily be somewhat larger than the poloidal field without affecting the dipole moment.

One possible observational consequence of magnetic splitting of frequency degeneracy for toroidal oscillations is the fact that it leaves m and -m modes degenerate to the extent that the field is axisymmetric about the magnetic axis. In contrast to the magnetic splitting between different $|m|$ modes (which appears to be on the order of several hertz and implies that the beat periods would be less than the rotation period), the m and -m modes would beat more slowly, and their beating could possibly be observed on a timescale of perhaps seconds to tens of seconds. The appearance of a particular frequency band beating on and off during the QPO activity could be a signature of the m and -m modes having [or, more precisely, of their symmetric and antisymmetric combinations having] slightly different frequencies. Such beating would be a measure of a non-axisymmetric component to the field. If, for example, the field is larger at $\phi = 0$ than at $\phi = \pi/2$, then the symmetric combination of the m=1 and m=-1 modes, proportional to cosϕ, would have a slightly higher frequency than the anti-symmetric mode, proportional to sinϕ.

The residual m,-m degeneracy could also be removed by rotation of a neutron star. For toroidal modes, the angular frequency shift due to rotation in a rotating reference frame attached to the star is $m\Omega/l(l+1)$ (see eg. Pekeris et al. (1961); Strohmayer (1991)), where Ω is the angular frequency of rotation. With period of rotation $P_{rot} \simeq 7.5s$ one can see that the splitting is quite small and modulations of the crustal displacement amplitude (beats) are possible with period $T_{mod} = P_{rot} l(l+1)/2m$. Since the physics of QPO variations in magnetar luminosity is somehow determined by crustal oscillations [e.g. Timokhin, Eichler & Lyubarsky (2007)], one might also expect time modulations in the observed QPO components on time scales of order 30 seconds at $l = m = 7$.

Another effect that could be looked for is the different damping rates of the symmetric and antisymmetric combinations due to coupling with the core. In general, such damping occurs when the frequency of the mode matches the resonant Alfven frequency of some connecting field lines in the core. But the damping rate would depend on the amplitude of the crustal oscillation at the latitude and longitudes where the frequency match happens to take place, and this amplitude can differ among the various linear combinations of symmetric modes. This suggests that QPO components could resolve to narrower frequency bands as the more rapidly damped linear combination gives way to the surviving combination. Detecting such an effect, however, would require good frequency resolution.

The differences between beating, resolution to the longest lived of several modes, and continuous frequency drift should all be made clear. We are unable to see how magnetic splitting or damping by the continuum leads to continuous frequency drift, and we see little if any evidence for it in the data of Strohmayer & Watts (2006).

The potential wealth of data available in magnetar seismology awaits confirmation of an accepted model for it. Future observations of intermediate flares, which may be more frequent than giant flares, may provide badly needed additional data.

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A. Vector spherical harmonics

Vector spherical harmonics are defined as

$$\vec{Y}^{L}_{JM}(\theta,\varphi) = \sum_{m,\sigma} C^{JM}_{Lm1\sigma} Y_{Lm}(\theta,\varphi)\, \vec{e}_{\sigma}$$

where

$$\vec{e}_{+1} = -\frac{1}{\sqrt{2}}\left(\vec{e}_x + i\ \vec{e}_y\right),\ \ \vec{e}_0 = \vec{e}_z\ ,\ \ \vec{e}_{-1} = \frac{1}{\sqrt{2}}\left(\vec{e}_x - i\ \vec{e}_y\right)$$

Here L can have values $L = J, J \pm 1$ for a given J .

During the calculations following formulae prove useful (for more information see e.g. (Varshalovich et al. 1988)).

$$\int d\Omega\ \vec{Y}^{L_1 *}_{J_1 M_1}\left(\theta,\varphi\right) \cdot \vec{Y}^{L_2}_{J_2 M_2}\left(\theta,\varphi\right) = \delta_{J_1 J_2}\delta_{L_1 L_2}\delta_{M_1 M_2}$$

$$\begin{aligned}\vec{Y}^{L_1}_{J_1 M_1}\left(\theta,\varphi\right) \times \vec{Y}^{L_2}_{J_2 M_2}\left(\theta,\varphi\right) &= i\sqrt{\frac{3}{2\pi}\left(2J_1+1\right)\left(2J_2+1\right)\left(2L_1+1\right)\left(2L_2+1\right)}\cdot \\ &\sum_{J,L}\left\{\begin{array}{ccc} J_1 & L_1 & 1 \\ J_2 & L_2 & 1 \\ J & L & 1 \end{array}\right\} C^{L0}_{L_1 0 L_2 0} C^{JM}_{J_1 M_1 J_2 M_2} \vec{Y}^{L}_{JM}\left(\theta,\varphi\right)\end{aligned}$$

$$\vec{\nabla} \times \left[f\left(r\right)\vec{Y}^{J+1}_{JM}\left(\theta,\varphi\right)\right] = i\sqrt{\frac{J}{2J+1}}\left(\frac{d}{dr} + \frac{J+2}{r}\right)\ f\left(r\right)\ \ \vec{Y}^{J}_{JM}\left(\theta,\varphi\right)$$

$$\begin{aligned}\vec{\nabla} \times \left[f\left(r\right)\vec{Y}^{J}_{JM}\left(\theta,\varphi\right)\right] &= i\sqrt{\frac{J}{2J+1}}\left(\frac{d}{dr} - \frac{J}{r}\right)\ f\left(r\right)\ \ \vec{Y}^{J+1}_{JM}\left(\theta,\varphi\right) \\ &+ i\sqrt{\frac{J+1}{2J+1}}\left(\frac{d}{dr} + \frac{J+1}{r}\right)\ f\left(r\right)\ \ \vec{Y}^{J-1}_{JM}\left(\theta,\varphi\right)\end{aligned}$$

$$\vec{\nabla} \times \left[f\left(r\right)\vec{Y}^{J-1}_{JM}\left(\theta,\varphi\right)\right] = i\sqrt{\frac{J+1}{2J+1}}\left(\frac{d}{dr} - \frac{J-1}{r}\right)\ f\left(r\right)\ \ \vec{Y}^{J}_{JM}\left(\theta,\varphi\right)$$

Now, taking into account that $\vec{u}^{(0)}_{lm}$ and $\vec{e}_z$ can be represented as [e.g. (Varshalovich et al. 1988)]

$$\vec{u}^{(0)}_{lm} = -i\sqrt{l(l+1)}w^{(0)}_l\left(r\right)\vec{Y}^{l}_{lm}, \quad \vec{e}_z = \sqrt{4\pi}\,\vec{Y}^{0}_{10}$$

one can use the above formulae to get the result (8,9).

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Table 1. Values of the function $F(l, m)$ for $l = 5, 7, 9$

m	$F(5, m)$	$F(7, m)$	$F(9, m)$
0	0.487	0.493	0.496
1	0.472	0.485	0.49
2	0.426	0.459	0.474
3	0.349	0.416	0.447
4	0.241	0.356	0.409
5	0.103	0.279	0.36
6	···	0.184	0.3
7	···	0.074	0.23
8	···	···	0.149
9	···	···	0.057